Journal of Statistical Physics, Vol. 75, Nos. 1/2, 1994

A Characterization of Gibbs States of Lattice Boson Systems

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Received August 6, 1993; final January 3, 1994

We consider lattice boson systems interacting via potentials which are superstable and regular. By using the Wiener integral formalism and the concept of conditional reduced density matrices we are able to give a characterization of Gibbs (equilibrium) states. It turns out that the space of Gibbs states is nonempty, convex, and also weak-compact if the interactions are of finite range. We give a brief discussion on the uniqueness of Gibbs states and the existence of phase transitions in our formalism.

KEY WORDS: Lattice boson systems; Wiener integral formalism; superstable interaction; Gibbs measures; Gibbs states; conditional reduced density matrix.

1. INTRODUCTION

The main purpose of this paper is to give a characterization of Gibbs states (equilibrium states) of lattice boson systems (lattice systems of quantum anharmonic oscillators) interacting via potentials which are superstable and regular. The model we consider can be viewed as a model for the quantum anharmonic crystals[(4)] and is closely related to lattice field theory with continuous times.[(1)] Recently the coupled lattice boson systems have been used to describe proton dynamics in hydrogen-bonded systems (ref. 16 and references therein). Thus it may be worth investigating the mathematical structure and physical properties of the systems.

There have been several studies on lattice boson systems. The infinite-volume limit of correlation functions for quantum anharmonic oscillators with gentle (bounded) perturbations has been studied in detail by Albeverio

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0022-4715/94/0400-0215$07.00/0

and Høegh-Krohn.[1] Park has constructed the infinite-volume limit theory of systems satisfying superstability and regularity conditions, and showed the clustering property of the Gibbs state in the high-temperature regime.[20] Recently the clustering property and analyticity of the infinite-volume-limit Gibbs state have been established for one-dimensional lattice boson systems for any value of nonzero temperature.[17] The existence of phase transitions has been proved for a class of quantum anharmonic oscillators.[12] Even if there were extensive studies as mentioned above, many problems concerning the detailed properties of the Gibbs states, such as an exact definition of equilibrium states, remain open. In this paper we give a characterization of Gibbs states and then investigate the structure of the space of Gibbs states for lattice boson systems. We then discuss the uniqueness of Gibbs states and the possibility of the first-order phase transitions in our formalism.

It is generally accepted that in quantum statistical mechanics equilibrium states are those of KMS states.[3,11,26,29] The algebra of observables is given by a quasilocal algebra $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}_\Lambda}$, where $\mathfrak{A}_\Lambda$ is the algebra of bounded linear operators on the local Hilbert space $\mathscr{H}_\Lambda$, $\Lambda \subset \mathbf{Z}^\nu$ (or $\mathbf{R}^\nu$), and the union runs over all bounded regions in $\mathbf{Z}^\nu$ (or $\mathbf{R}^\nu$). Let τ_t, $t \in \mathbf{R}$, be a one-parameter group of time evolution automorphisms on $\mathfrak{A}$. For a state on $\mathfrak{A}$ to be an equilibrium state one demands it to satisfy a typical condition, the KMS condition: a state ρ on a C^*-dynamical system $(\mathfrak{A}, \tau)$ is a τ-KMS state at $\beta \in \mathbf{R}$ if $\rho(A\tau_{i\beta}(B)) = \rho(BA)$ for all A, B in a suitable norm dense subalgebra of $\mathfrak{A}$. Thus, in order to construct the infinite-volume theory for a given model, one has to construct the time evolution automorphism τ and a state ρ from local time automorphisms

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \qquad A \in \mathfrak{A}_\Lambda$$

and the local Gibbs state

$$\rho_\Lambda(A) = Z_\Lambda^{-1} \operatorname{Tr}(e^{-\beta H_\Lambda} A)$$

respectively, where H_Λ is a local Hamiltonian and Z_Λ is the normalization factor. For details we refer to ref. 3. The above framework has been applied successively to bounded quantum lattice spin systems.[3,11]

In the cases of lattice boson systems and (interacting) quantum particle systems, the situations are much more complex.[3] A global statement of the time evolution as a group of automorphisms of the quasilocal algebra has not been obtained. The problem with the time evolution has been partially circumvented by using the Green's function method.[19,20] The finite-volume Green's functions are given by

$$G_\Lambda(A, B; t) = \rho_\Lambda(A\tau_t^\Lambda(B))$$

Under appropriate conditions on the interactions, the infinite-volume-limit Green's functions exist and satisfy desired properties. From the infinite-volume Green's functions it was possible to construct a Hilbert space $\mathfrak{R}$, a strongly continuous unitary representation U of the time evolution on $\mathfrak{R}$, and the algebra of observables. For detailed discussions, we refer to refs. 3, 19, and 20. In the framework of the Green's function method it is difficult to give a characterization of Gibbs states as states on the quasilocal algebra $\mathfrak{A}$.

Recently Fichtner and Freudenberg[5 7] tried to characterize the normal (locally normal) states for quantum particle systems by using a point process and conditional reduced density matrices (CRDM). The point process contains information connected with position measurements and CRDM with local Gibbs states (local density matrices). Using the concept of CRDM together with a process, we would like to characterize Gibbs states for lattice boson systems and then investigate the structure of the Gibbs state space. More precisely, for any bounded $\Lambda \subset \mathbf{Z}^\nu$ let $\mathfrak{A}_\Lambda$ be the algebra of bounded linear operators on $\mathscr{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathscr{H}_i$, where $\mathscr{H}_i$ is the copy of the *a priori* Hilbert space $L^2(\mathbf{R}^d)$. The local Hamiltonians are of the form

$$H_\Lambda = -\tfrac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda)$$

where Δ_i is the Laplacian operator on $L^2(\mathbf{R}^d)$ and $V(x_\Lambda)$ is an interaction function on $(\mathbf{R}^d)^\Lambda$. By the Feynman–Kac formula, the operator $\exp(-H_\Lambda)$ has its integral kernel

$$e^{-H_\Lambda}(x_\Lambda, y_\Lambda) = \int P_{x_\Lambda, y_\Lambda}(ds_\Lambda)\, e^{-V(s_\Lambda)}$$

where x_Λ and y_Λ are points in $(\mathbf{R}^d)^\nu$, $s_\Lambda \in (C[0, 1])^\Lambda$, $V(s_\Lambda) = \int_0^1 V(s_\Lambda(t))\, dt$, and P_{x_Λ, y_Λ} is the conditional Wiener measure (see Section 2 for the notation). We thus have an integration on the path space $\Omega_\Lambda = (C[0, 1])^\Lambda$. This gives us a hint to introduce a family of conditional states (specifications) on each subalgebra $\mathfrak{A}_\Lambda$ and Gibbs measures on $\Omega = (C[0, 1])^{\mathbf{Z}^\nu}$. For a state ρ on $\mathfrak{A}$ to be a Gibbs state we demand that it satisfy the condition

$$\rho(A) = \int d\nu(\bar{s})\, \rho_\Lambda^{\bar{s}}(A), \qquad A \in \mathfrak{A}_\Lambda$$

for some Gibbs measure ν on Ω (together with some additional conditions), where $\rho_\Lambda^{\bar{s}}$ is the conditional local state defined by the conditional reduced density matrix [see Definition 2.8 and the expression (3.2)]. Under the superstability and regularity conditions it turns out that the space of Gibbs state is nonempty, convex, and weak*-compact. Thus we are able to give sufficient conditions for the uniqueness of Gibbs states and the existence of phase transitions in our formalism. For detailed discussions, see Sections 2 and 3.

We organize this paper as follows: In Section 2, we introduce notations, definitions, and basic assumptions on the potentials together with some necessary preliminaries on the Wiener measure. We then give a characterization of Gibbs states in terms of Gibbs measures and conditional reduced density matrices, and then state our main results. In Section 3 the proofs of the main results are produced. In Section 4 we give a brief discussion on the uniqueness of Gibbs states and the existence of phase transitions. In Section 5 we list open problems related to further properties of Gibbs states and dynamics (time evolutions) of the systems. In an appendix we give proofs of technical estimates (superstability estimates).

2. PRELIMINARIES AND MAIN RESULTS

We consider lattice boson systems on the lattice $\mathbf{Z}^\nu$. By $\mathscr{C}$ we mean the class of finite subsets of $\mathbf{Z}^\nu$. At each site $i \in \mathbf{Z}^\nu$ we associate an identical copy of the Hilbert space $L^2(\mathbf{R}^d, dx)$, where dx is the Lebesgue measure on $\mathbf{R}^d$. For $x = (x^1,..., x^d) \in \mathbf{R}^d$, $i = (i_1,..., i_\nu) \in \mathbf{Z}^\nu$ we write

$$|x| = \left[\sum_{i=1}^{d} (x^i)^2\right]^{1/2}, \qquad |i| = \max_{1 \leqslant l \leqslant \nu} |i_l| \tag{2.1}$$

For a bounded region $\Lambda \subset \mathbf{Z}^\nu$ we write

$$x_\Lambda = \{x_i : i \in \Lambda\}, \qquad dx_\Lambda = \prod_{i \in \Lambda} dx_i \tag{2.2}$$

The Hilbert space for lattice boson systems in $\Lambda \in \mathscr{C}$ is given by

$$\begin{aligned} \mathscr{H}_\Lambda &= \bigotimes_{i \in \Lambda} L^2(\mathbf{R}^d, dx_i) \\ &= L^2((\mathbf{R}^d)^\Lambda, dx_\Lambda) \end{aligned} \tag{2.3}$$

We introduce a Hamiltonian operator on $\mathscr{H}_\Lambda$ by

$$\begin{aligned} H_\Lambda &= -\tfrac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda) \\ V(x_\Lambda) &\equiv \sum_{\Delta \subset \Lambda} \Phi_\Delta(x_\Delta) \end{aligned} \tag{2.4}$$

where Δ_i is the Laplacian operator for the variable $x_i \in \mathbf{R}^d$ and for each $\Delta \subset \mathbf{Z}^\nu$, Φ_Δ is the interaction potential which is a measurable real-valued function on $(\mathbf{R}^d)^\Delta$.

Throughout this paper we impose the following conditions on the interaction:

Assumption 2.1. The interaction $\Phi = (\Phi_\Delta)_{\Delta \subset \mathbf{Z}^\nu}$ satisfies the following conditions:

(a) Φ_Δ is a Borel measurable function on $(\mathbf{R}^d)^\Delta$.

(b) Φ_Δ is invariant under translations of $\mathbf{Z}^\nu$.

(c) (Superstability). There are $A > 0$ and $c \in \mathbf{R}$ such that for every $x_\Lambda \in (\mathbf{R}^d)^\Lambda$,

$$V(x_\Lambda) = \sum_{\Delta \subset \Lambda} \Phi_\Delta(x_\Lambda) \geqslant \sum_{i \in \Lambda} [A x_i^2 - c]$$

(d) (Strong regularity). There exists a decreasing positive function Ψ on the natural integers such that

$$\Psi(r) \leqslant K r^{-\nu-\varepsilon} \qquad \text{for some} \quad K \text{ and } \varepsilon > 0 \quad \text{with} \quad \sum_{i \in \mathbf{Z}^\nu} \Psi(|i|) < A$$

Furthermore, if Λ_1, Λ_2 are disjoint finite subsets of $\mathbf{Z}^\nu$ and if one writes

$$V(x_{\Lambda_1 \cup \Lambda_2}) = V(x_{\Lambda_1}) + V(x_{\Lambda_2}) + W(x_{\Lambda_1}, x_{\Lambda_2})$$

then the bound

$$|W(x_{\Lambda_1}, x_{\Lambda_2})| \leqslant \sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \Psi(|i-j|) \tfrac{1}{2}(x_i^2 + x_j^2)$$

holds.

Remark. The first part of Assumption 2.1(d) $[\sum_{i \in \mathbf{Z}^\nu} \Psi(|i|) < A]$ can be weakened as in Hypothesis 4.1 of ref. 15.

For a bounded domain $\Lambda \subset \mathbf{Z}^\nu$, the C^*-algebra of local observables is defined by

$$\mathfrak{A}_\Lambda = \mathscr{L}(\mathscr{H}_\Lambda) \tag{2.5}$$

where $\mathscr{L}(\mathscr{H}_\Lambda)$ is the algebra of all bounded operators on $\mathscr{H}_\Lambda$. If $\Lambda_1 \cap \Lambda_2 = \varnothing$, then $\mathscr{H}_{\Lambda_1 \cup \Lambda_2} = \mathscr{H}_{\Lambda_1} \otimes \mathscr{H}_{\Lambda_2}$, and $\mathfrak{A}_{\Lambda_1}$ is isomorphic to the

C^*-algebra $\mathfrak{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$, where $\mathbf{1}_{\Lambda_2}$ denotes the identity operator on $\mathscr{H}_{\Lambda_2}$. In this way we identify $\mathfrak{A}_\Lambda$ as a subalgebra on $\mathfrak{A}_{\Lambda'}$ if $\Lambda \subset \Lambda'$. Let

$$\mathfrak{A} = \overline{\bigcup_{\Lambda \in \mathscr{C}} \mathfrak{A}_\Lambda} \tag{2.6}$$

be the algebra of the quasilocal observables. Notice that $\mathfrak{A}$ has an identity.

As mentioned in Introduction, the Wiener integral formalism of lattice boson systems plays a key role in the sequel. For notational convenience we set the inverse temperature β to be one.

For $x, y \in \mathbf{R}^d$ let us denote by $W_{x,y}$ the set of continuous paths ω: $[0, 1] \to \mathbf{R}^d$ with $\omega(0) = x$, $\omega(1) = y$. $W_{x,y}$ is endowed with the standard Borel space structure. By $P_{x,y}$ we mean the conditional Wiener measure on $W_{x,y}$,[10,28]

$$P_{x,y}(W_{x,y}) = (2\pi)^{-d/2} \exp(-1/2\,|x-y|^2)$$

For finite $\Lambda \subset \mathbf{Z}^\nu$ and $x_\Lambda, y_\Lambda \in (\mathbf{R}^d)^\Lambda$, W_{x_Λ, y_Λ} and P_{x_Λ, y_Λ} mean the Cartesian product $\mathsf{X}_{j \in \Lambda} W_{x_j, y_j}$ and the product measure $\mathsf{X}_{j \in \Lambda} P_{x_j, y_j}$, respectively. We identify the space $W_{x,x}$, $x \in \mathbf{R}^d$, with a single space $W = W_{0,0}$ by means of the mapping $\omega \leftrightarrow \omega + x$. Measures $P_{x,x}$ and $P = P_{0,0}$ are transformed thereby into each other. Furthermore we use the map $W_{x,y} \leftrightarrow W_{0,0}$ given by $\omega \leftrightarrow \omega + L_{x,y}$, where $L_{x,y}$ is the linear function $L_{x,y}(t) = x + t(y-x)$. The measure $P_{x,y}$ is transformed thereby into $\exp(-\frac{1}{2}|x-y|^2)\, P_{0,0}$. The product space W_{x_Λ, y_Λ} is transformed into W^Λ analogously in which the function $L_{x_\Lambda, y_\Lambda}(t) = x_\Lambda + t(y_\Lambda - x_\Lambda)$ is used. We shall use the notation

$$S = \mathbf{R}^d \times W \qquad \text{and} \qquad s = (x, \omega) \in S \tag{2.7}$$

as well as

$$\hat{S} = \mathbf{R}^d \times \mathbf{R}^d \times W \qquad \text{and} \qquad \hat{s} = (x, y, \omega) \in \hat{S} \tag{2.8}$$

The spaces S and $\hat{S}$ are provided with the norms

$$\|s\| = \left[\int_0^1 |s(t)|^2\, dt\right]^{1/2} \tag{2.9}$$

and

$$\|\hat{s}\| = \left[\int_0^1 |\hat{s}(t)|^2\, dt\right]^{1/2} \tag{2.10}$$

where $s(t) = x + \omega(t)$, $\hat{s}(t) = L_{x,y}(t) + \omega(t)$. If there is no confusion we will just write s^2, $\hat{s}^2$ for $\|s\|^2$, $\|\hat{s}\|^2$, respectively. We give an *a priori* measure λ on S as follows:

$$\lambda(ds) \equiv dx\, P(d\omega), \qquad s = (x, \omega)$$

where dx is the Lebesgue measure on $\mathbf{R}^d$. We note that

$$\int \lambda(ds)\, e^{-\alpha s^2} < \infty, \qquad s = (x, \omega) \tag{2.11}$$

for all $\alpha > 0$. In fact,

$$\begin{aligned}\int \lambda(ds) \exp(-\alpha s^2) &= \int dx \int P_{x,x}(d\omega) \exp\left[-\alpha \int_0^1 \omega^2(t)\, dt\right] \\ &= \operatorname{Tr}(\exp[-(-\tfrac{1}{2}\Delta + \alpha x^2)]) < \infty\end{aligned}$$

Given a finite $\Lambda \subset \mathbf{Z}^\nu$, s_Λ and $\hat{s}_\Lambda$ have the obvious meaning. We use the notation

$$\Omega \equiv S^{\mathbf{Z}^\nu} = (\mathbf{R}^d \times W)^{\mathbf{Z}^\nu} \tag{2.12}$$

and

$$\hat{\Omega} \equiv \hat{S}^{\mathbf{Z}^\nu} = (\mathbf{R}^d \times \mathbf{R}^d \times W)^{\mathbf{Z}^\nu} \tag{2.13}$$

For each $i \in \mathbf{Z}^\nu$, let $p_i \colon \Omega \to S$ be the projection, $p_i(s) = s_i$, the value (path) on the ith site. For each subset $\Lambda \subset \mathbf{Z}^\nu$, we have a local σ-algebra $\mathscr{F}_\Lambda$, which is the minimal σ-algebra for which p_i, $i \in \Lambda$, are measurable. We simply write $\mathscr{F}$ for $\mathscr{F}_{\mathbf{Z}^\nu}$. By $\mathscr{P}(\Omega, \mathscr{F})$ we mean the probability measures on Ω.

Before introducing Gibbs measures on Ω, we give the notion of regular measures on Ω:

Definition 2.2. A Borel probability measure on $(\Omega, \mathscr{F})$ is said to be regular if there exist $\bar{A} > 0$ and $\bar{\delta}$ so that the projection $\mu(ds_\Lambda)$ on any $(\Omega, \mathscr{F}_\Lambda)$ satisfies

$$g(s_\Lambda \mid \mu) \leqslant \exp\left[-\sum_{i \in \Lambda} (\bar{A} s_i^2 - \bar{\delta})\right]$$

where $g(s_\Lambda \mid \mu)$ is such that $\mu(ds_\Lambda) = \lambda(ds_\Lambda)\, g(s_\Lambda \mid \mu)$.

We write that

$$\begin{aligned}\Phi_\Delta(s_\Delta) &= \int_0^1 \Phi_\Delta(s_\Delta(t))\, dt \\ V(s_\Lambda) &= \sum_{\Delta \subset \Lambda} \Phi_\Delta(s_\Delta)\end{aligned} \tag{2.14}$$

and for $s \in \Omega$ and $\Lambda \subset \mathbf{Z}^\nu$,

$$W(s_\Lambda, s_{\Lambda^c}) = \sum_{\substack{\Delta \cap \Lambda \neq \varnothing \\ \Delta \cap \Lambda^C \neq \varnothing}} \Phi_\Delta(s_\Delta) \tag{2.15}$$

Let us define

$$\begin{aligned} \mathfrak{S} &= \bigcup_{N \in \mathbf{N}} \mathfrak{S}_N \\ \mathfrak{S}_N &= \left\{ s \in \Omega \colon \forall l, \sum_{|i| \leqslant l} s_i^2 \leqslant N^2(2l+1)^\nu \right\} \end{aligned} \tag{2.16}$$

This definition is invariant under linear translations of $\mathbf{Z}^\nu$. Following the method used in the proof of the first part of Proposition 5.2 of ref. 26, it is easy to show that each regular measure on $(\Omega, \mathscr{F})$ has its support on $\mathfrak{S}$. We say that such a measure is *tempered.*[26]

The partition function in a finite $\Lambda \subset \mathbf{Z}^\nu$ for the interaction Φ with boundary condition $\bar{s} \in \mathfrak{S}$ is defined by

$$Z_\Lambda^\Phi(\bar{s}) \equiv \int \lambda^\Lambda(ds_\Lambda) \exp[-V(s_\Lambda) - W(s_\Lambda, \bar{s}_{\Lambda^c})] \tag{2.17}$$

We note that the partition function is well defined from the assumptions for Φ. The Gibbs specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda \in \mathscr{C}}$ with respect to $\mathfrak{S}$ is defined by[9,23]

$$\gamma_\Lambda^\Phi(A \mid \bar{s}) = \begin{cases} Z_\Lambda^\Phi(\bar{s})^{-1} \int \lambda^\Lambda(ds_\Lambda) \exp[-V(s_\Lambda) - W(s_\Lambda, \bar{s}_{\Lambda^c})] \\ \quad \times 1_A(s_\Lambda \bar{s}_{\Lambda^c}) \qquad \text{if} \quad \bar{s} \in \mathfrak{S} \\ 0 \qquad\qquad\qquad\quad \text{if} \quad \bar{s} \notin \mathfrak{S} \end{cases} \tag{2.18}$$

where $A \in \mathscr{F}$ and 1_A is the indicator function on A and $s_\Lambda \bar{s}_{\Lambda^c}$ is the configuration defined by s_Λ on Λ and $\bar{s}_{\Lambda^c}$ on Λ^C, respectively. It is easy to check that the Gibbs specification satisfies the consistency condition[9,23]: For $\Delta \subset \Lambda$, $\bar{s} \in \mathfrak{S}$,

$$\begin{aligned} \gamma_\Lambda^\Phi \gamma_\Delta^\Phi(A \mid \bar{s}) &\equiv \int_{\mathfrak{S}} \gamma_\Lambda^\Phi(ds^* \mid \bar{s})\, \gamma_\Delta^\Phi(A \mid s^*) \\ &= \gamma_\Lambda^\Phi(A \mid \bar{s}) \end{aligned}$$

We now give a definition of Gibbs measures on $(\Omega, \mathscr{F})$:

Definition 2.3. A Gibbs measure μ for the potential Φ is a tempered Borel probability measure on $(\Omega, \mathscr{F})$ satisfying the equilibrium equations

$$\mu(A) = \int \mu(d\bar{s})\, \gamma_\Lambda^\Phi(A \mid \bar{s}), \qquad A \in \mathscr{F}$$

We denote by $\mathscr{G}^\Phi(\Omega)$ the family of all Gibbs measures.

We topologize the space $\mathscr{P}(\Omega, \mathscr{F})$ with the topology of local convergence[9,21]: For each $\mu \in \mathscr{P}(\Omega, \mathscr{F})$ the sets

$$\{\nu \in \mathscr{P}(\Omega, \mathscr{F}) \colon \max_{1 \leqslant k \leqslant n} |\nu(A_k) - \mu(A_k)| < \varepsilon\}$$

with $A_1, \dots, A_n \in \bigcup_{\Lambda \in \mathscr{C}} \mathscr{F}_\Lambda$, $\varepsilon > 0$, and $n \geqslant 1$ form a base of neighborhoods of μ. If the potential Φ satisfies the conditions in Assumption 2.1, then it turns out that $\mathscr{G}^\Phi(\Omega)$ is nonempty, closed, convex, compact, and has Choquet simplex structure.

Before studying the structure of the Gibbs measures let us introduce the notion of boundary conditions.[14,15]

Definition 2.4. (Boundary conditions). We consider free, pure, and general boundary conditions, and for all $\Lambda \in \mathscr{C}$, they give probability measures on $(\Omega, \mathscr{F})$ as follows: For any $A \in \mathscr{F}$:

(a) Free b.c.:

$$\nu_\Lambda^{(0)}(A) = Z_\Lambda^{(0)^{-1}} \int \lambda(ds_\Lambda) \exp[-V(s_\Lambda)]\, 1_A(s_\Lambda o_{\Lambda^c})$$

($Z_\Lambda^{(0)}$ is a normalization constant and o_{Λ^c} means the zero configuration on Λ^C).

(b) Pure b.c.:

$$\nu_\Lambda^{(\bar{s})}(A) = \gamma_\Lambda(A \mid \bar{s}) \qquad (\bar{s} \in \mathfrak{S})$$

(c) General b.c.:

$$\nu_\Lambda^{(\xi)}(A) = \int \xi(d\bar{s})\, \gamma_\Lambda(A \mid \bar{s})$$

(ξ is a regular measure).

For $\Delta \subset \Lambda$, $\bar{s} \in \mathfrak{S}$, let us put

$$g_\Lambda(s_\Delta \mid \bar{s}) \equiv Z_\Lambda(\bar{s})^{-1} \int \lambda(ds_{\Lambda \setminus \Delta}) \exp[-V(s_\Lambda) - W(s_\Lambda, \bar{s}_{\Lambda^c})] \quad (2.19)$$

We have a Ruelle-type probablity estimate.[14,15,26,27]

Proposition 2.5. Let the hypotheses of Assumption 2.1 hold. Then there exist $A^*>0$ and δ such that for every Δ and $\bar{s}\in\mathfrak{S}$ there exists $\Lambda(\Delta,\bar{s})$ such that whenever $\Lambda\supset\Lambda(\Delta,\bar{s})$ the bound

$$g_\Lambda(s_\Delta\mid\bar{s})\leqslant\exp\left[-\sum_{i\in\Delta}(A^*s_i^2-\delta)\right]$$

holds.

The proof of the above proposition is given in the appendix. Using the above probability estimate, we have the following result.

Theorem 2.6. The net $(\nu_\Lambda^{(*)})_{\Lambda\in\mathscr{C}}$ (by (*) we mean any b.c.) has a cluster point in $\mathscr{G}^\Phi(\Omega)$.

On the other hand, we have also a converse result to the above theorem: a tempered equilibrium (Gibbs) measure is necessarily regular. In fact we have:

Theorem 2.7. Let the hypotheses of Assumption 2.1 hold. Then any Gibbs measure $\nu\in\mathscr{G}^\Phi(\Omega)$ is regular. Futhermore, $\mathscr{G}^\Phi(\Omega)$ is nonempty, convex, compact in the local convergence topology, and a Choquet simplex.

As a consequence ν is an infinite-volume Gibbs measure with general b.c. detemined by a regular measure which is just ν. Therefore ν is the (trivial) limit of general b.c. Gibbs measures.[14,15] We postpone the proofs of Theorem 2.6 and Theorem 2.7 to the next section.

Let us now consider Gibbs (equilibrium) states on the quasilocal algebra $\mathfrak{A}$. For a finite set $\Lambda\subset\mathbf{Z}^\nu$ and a configuration $\bar{s}\in\mathfrak{S}$, we define a function $k_\Lambda(x_\Lambda,y_\Lambda;\bar{s})$, $x_\Lambda,y_\Lambda\in(\mathbf{R}^d)^\Lambda$, which takes the role of the conditional reduced density matrix[3,6,7]:

$$k_\Lambda(x_\Lambda,y_\Lambda;\bar{s})\equiv Z_\Lambda(\bar{s})^{-1}\exp(-\tfrac{1}{2}|x_\Lambda-y_\Lambda|^2)\int P(d\omega_\Lambda)$$
$$\times\exp[-V(\omega_\Lambda+L_{x_\Lambda,y_\Lambda})-W(\omega_\Lambda+L_{x_\Lambda,y_\Lambda},\bar{s}_{\Lambda^c})]\qquad(2.20)$$

Remember that $P(d\omega_\Lambda)$ is the conditional Wiener measure on $W_{0,0}$ and $L_{x_\Lambda,y_\Lambda}(t)=x_\Lambda+t(y_\Lambda-x_\Lambda)$. With the help of these functions and the regular Gibbs measures we define Gibbs states as follows:

Definition 2.8. We say that a state ρ on the quasilocal algebra $\mathfrak{A}=\overline{\bigcup_{\Lambda\in\mathscr{C}}\mathfrak{A}_\Lambda}$ is a Gibbs state if the restriction ρ_Λ of ρ to $\mathfrak{A}_\Lambda$ is given by

$$\rho_\Lambda(A)=\mathrm{Tr}_{\mathscr{H}_\Lambda}(K_\Lambda A),\qquad A\in\mathfrak{A}_\Lambda$$

where the density matrix K_Λ is defined by its integral kernel

$$K_\Lambda(x_\Lambda, y_\Lambda) = \int \nu(d\bar{s})\, k_\Lambda(x_\Lambda, y_\Lambda; \bar{s})$$

with a Gibbs measure $\nu \in \mathscr{G}^\Phi(\Omega)$.

Remark. In the definition of Gibbs state we require the positivity of K_Λ. In general the function $K_\Lambda(x_\Lambda, y_\Lambda)$ may not be a positive-definite function. But if the Gibbs measure ν is an infinite-volume limit of local Gibbs states $\nu_\Lambda^{(\bar{s})}$ with pure b.c. $\bar{s} \in \mathfrak{S}$ of symmetric paths ($s \in \mathfrak{S}$ is defined to be symmetric if $s(t) = s(1-t)$, $t \in [0, 1]$), then it can be shown that $K_\Lambda(x_\Lambda, y_\Lambda)$ is a positive-definite function.

The interaction Φ is said to be of finite range with interaction range $R \in \mathbf{R}$ if $\Phi_\Delta = 0$ for all Δ with $\mathrm{dia}(\Delta) > R$.

Theorem 2.9. For a given interaction Φ, let $\mathscr{G}^\Phi(\mathfrak{A})$ be the family of all Gibbs states on $\mathfrak{A}$. If the interaction Φ satisfies the hypotheses of Assumption 2.1, then $\mathscr{G}^\Phi$ is nonempty, convex, and also weak*-compact if the interaction is of finite range.

The finite-range assumption on Φ for the weak*-compactness property should be removed. We impose it because of a technical difficulty in the proof. We will show Theorem 2.9 in the next section. It may be worth giving some questions related to further properties of Gibbs states.

Remark. (a) Is any $\rho \in \mathscr{G}^\Phi(\mathfrak{A})$ a modular state?[3] If it is so, one has the modular automorphisms. Using the Green's function method,[20] it can be shown that any infinite-volume-limit Gibbs state with free (or constant) b.c. is a modular state.

(b) Is $\mathscr{G}^\Phi(\mathfrak{A})$ a Choquet simplex? If each $\rho \in \mathscr{G}^\Phi(\mathfrak{A})$ is a modular state, then the answer is positive by the KMS conditions with respect to the modular automorphisms and the method used in refs. 3 and 11.

For the integral kernel of the density matrix we have the following bound:

Proposition 2.10. Let the interaction Φ satisfy the hypotheses of Assumption 2.1. Then there exist positive numbers c, d, and e such that the bound

$$K_\Lambda(x_\Lambda, y_\Lambda) \leqslant \exp\left[-\sum_{i \in \Lambda} (c x_i^2 + d y_i^2 - e) \right]$$

holds. Furthermore, if we assume that the interaction is of any polynomial type, then for each $n \in \mathbf{N}$, $i \in \Lambda$, $K_\Lambda(x_\Lambda, y_\Lambda)$ is n-times differentiable and the bound

$$|\partial^n_{x_i} K_\Lambda(x_\Lambda, y_\Lambda)_{x_\Lambda = y_\Lambda}| \leqslant \exp\left[-\sum_{i \in \Lambda} (c' x_i^2 - e')\right]$$

holds for some positive c' and e'.

The above bounds should be useful in the further study of the model. We will produce the proof of the above proposition in the appendix.

3. PROOFS OF MAIN RESULTS

Before proving the main results we remark that our formalism has been set up in such a way that most of the results of Gibbs measures on $(\Omega, \mathscr{F})$ can be proven by using the methods employed in refs. 14, 15, 17, 19, 25, and 26 with necessary modifications. In particular we closely follow the contents of Section 4 (and its Appendix) of ref. 14 to establish basic properties of the Gibbs measures.

In this section we give proofs for the main results in Section 2. The technical estimates in Propositions 2.5 and 2.10 are be carried out in the appendix.

To prove Theorem 2.6 we need the following result:

Lemma 3.1. (a) Let $\xi \in \mathscr{P}(\Omega, \mathscr{F})$ be a regular measure. Then for all $\varepsilon > 0$, $\exists N_0$ such that $\xi(\mathfrak{S}^C_{N_0}) < \varepsilon$ (and hence $\forall N \geqslant N_0$).

(b) There exist $A^* > 0$ and $\delta > 0$ such that the following holds: For every Δ and $N_0 \in \mathbf{N}$ there is $\Lambda(\Delta, N_0)$ such that the bound

$$g_\Lambda(s_\Delta \mid \bar{s}) \leqslant \exp\left[-\sum_{i \in \Delta} (A^* s_i^2 - \delta)\right] \qquad \text{for} \quad \Lambda \supset \Lambda(\Delta, N_0)$$

holds uniformly for $\bar{s} \in \mathfrak{S}_{N_0}$, where g_Λ is defined in (2.19).

Proof. (a) The inequality follows from the fact that ξ is a regular measure (and hence ξ is supported on $\mathfrak{S} = \bigcup_N \mathfrak{S}_N$) and that the sequence $(\mathfrak{S}_N)_{N \in \mathbf{N}}$ is increasing.

(b) Let us fix N_0 and Δ (we may assume Δ contains the origin). There exists $N_1 > N_0$ such that $\sum_{i \in [n]^*} \bar{s}_i^2 \leqslant N_1^2 V_n$ for all $\bar{s} \in \mathfrak{S}_{N_0}$ and $n \in \mathbf{N}$, where $[n]^*$ is any translation of $[n]$ coming from a translation of the origin within Δ and V_n is the volume of $[n]$ (see the appendix for the notation). We note that in the proof of Proposition 2.5 in the appendix the set $\Lambda(\Delta, \bar{s})$ ($\bar{s} \in \mathfrak{S}_{N_0}$) depends on such a number N_1 and not on the individual configurations $\bar{s} \in \mathfrak{S}_{N_0}$. Hence the bound in the lemma holds uniformly for $\bar{s} \in \mathfrak{S}_{N_0}$ if $\Lambda \supset \Delta$ is sufficiently large. ▮

Proof of Theorem 2.6. We consider the case of general b.c. Other cases can be dealt with in a similar manner. Let $\xi \in \mathscr{P}(\Omega, \mathscr{F})$ be a regular measure and $(\nu_\Lambda^{(\xi)})_{\Lambda \in \mathscr{C}}$ be the net of associated probability measures on $(\Omega, \mathscr{F})$. By Proposition 4.9 of ref. 9, the existence of a cluster point will be shown just after we show that $(\nu_\Lambda^{(\xi)})$ is locally equicontinuous, i.e., given a sequence $(A_m) \in \mathscr{F}_\Delta$ with $A_m \downarrow \varnothing$,

$$\lim_{m \to \infty} \limsup_{\Lambda \in \mathscr{C}} \nu_\Lambda^{(\xi)}(A_m) = 0 \tag{3.1}$$

Now let $(A_m) \in \mathscr{F}_\Delta$, $A_m \downarrow \varnothing$ be given. Then

$$\nu_\Lambda^{(\xi)}(A_m) = \int_{\mathfrak{S}} \xi(d\bar{s})\, Z_\Lambda(\bar{s})^{-1} \int \lambda(ds_\Lambda) \exp[-V(s_\Lambda) - W(s_\Lambda, \bar{s}_{\Lambda^c})]\, 1_{A_m}(s_\Delta)$$

$$\leqslant \varepsilon + \int_{\mathfrak{S}_{N_0}} \xi(d\bar{s}) \int \lambda(ds_\Delta)\, 1_{A_m}(s_\Delta)\, g_\Lambda(s_\Delta \mid \bar{s})$$

Equation (3.1) follows from Lemma 3.1, the Lebesgue dominated convergence theorem, and the arbitrariness of ε (in that order). The proof that the limiting measure is Gibbsian can be carried out in the same way as in the proof of Theorem 4.5 of ref. 14. We remark that the proof of the Gibbs property for limiting measures in ref. 14 relies on the basic estimates in Lemma A4.1 of ref. 14 which are direct consequences of Theorem 4.1 of ref. 14. We derive the result corresponding to Theorem 4.1 of ref. 14 in the appendix (Lemma A.4) and so we can follow the argument used in ref. 14 to prove our result. Even if the proof in ref. 14 is given only for the case of pure boundary conditions (which differ from ours), a similar argument can be extended to general boundary conditions as mentioned in ref. 14. ▮

Proof of Theorem 2.7. The regularity of the Gibbs measure is given in Lemma A.2 in the appendix. The nonemptiness and convexity are obvious. To prove compactness, given a net $(\nu_\alpha) \in \mathscr{G}^\Phi(\Omega)$ of Gibbs measures, we rely again on showing that (ν_α) is locally equicontinuous. Let $(A_m) \in \mathscr{F}_\Delta$ be such that $A_m \downarrow \varnothing$. Using the equilibrium conditions and regularity of Gibbs measures in Lemma A.2, we obtain that

$$\nu_\alpha(A_m) = \int \lambda(ds_\Delta)\, 1_{A_m}(s_\Delta)\, g(s_\Delta \mid \nu_\alpha)$$

$$\leqslant \int \lambda(ds_\Delta)\, 1_{A_m}(s_\Delta) \exp\left[-\sum_{i \in \Delta} (\bar{A} s_i^2 - \delta)\right]$$

uniformly in A_α. Thus the local equicontinuity of a net (ν_α) follows from the above bound.

Consider the linear space $\mathscr{L}$ of real measures on $(\Omega, \mathscr{F})$ which are tempered and satisfy the equilibrium conditions. Denote by $\mathscr{S}$ the intersection of the cone $\mathscr{K}$ of positive measures in $\mathscr{L}$ with the hyperplane $\{\mu \mid \mu(1)=1\}$. Let γ_Δ be the Gibbs specifications defined in (2.18). For any $A \in \mathscr{F}$, $\gamma_\Delta(A \mid \bar{s}) \geqslant 0$. Using this fact and the equilibrium conditions, it is easy to show that if $\mu \in \mathscr{L}$, then $|\mu| \in \mathscr{L}$. With respect to the usual order on measure any two elements μ_1 and μ_2 have a l.u.b. $[(\mu_1+\mu_2)+|\mu_1-\mu_2|]/2$ and a g.l.b. $[(\mu_1+\mu_2)-|\mu_1-\mu_2|]/2$, which are again in $\mathscr{L}$. Since $\mathscr{L}$ is a lattice for order defined by $\mathscr{K}$, $\mathscr{S}$ $[=\mathscr{G}^\Phi(\Omega)]$ is a simplex. The proof of Theorem 2.7 is now completed. ∎

Proof of Theorem 2.9. The existence of a Gibbs state follows immediately from Theorem 2.6 (and the Remark below it). The convexity also follows from Theorem 2.6. To prove that $\mathscr{G}^\Phi(\mathfrak{A})$ is weak*-compact it is sufficient to show that $\mathscr{G}^\Phi(\mathfrak{A})$ is weak*-closed by the Banach–Alaoglu theorem. We prove the weak*-closedness of $\mathscr{G}^\Phi(\mathfrak{A})$ as follows: Let (ρ_α) be a net of Gibbs states which converges to $\hat{\rho}$ in the weak*-topology. By the Gibbs condition (Definition 2.8) there exists a net (ν_α) of Gibbs measures which converges to a Gibbs measure $\hat{\nu}$ in the topology of local convergence. We must show that $\hat{\nu}$ defines $\hat{\rho}$ through the Gibbs condition. Denote by $\mathfrak{T}_\Delta^{(2)}$ the family of Hilbert–Schmidt operators in $\mathfrak{A}_\Delta$. Notice that since $\mathfrak{T}_\Delta^{(2)}$ is σ-weakly dense in $\mathfrak{A}_\Delta$ (von Neumann density theorem), any normal state on $\mathfrak{A}_\Delta$ is uniquely defined by its restriction on $\mathfrak{T}_\Delta^{(2)}$. By Proposition 2.10 and the Fubini theorem, any Gibbs state can be written as

$$\rho(A)=\int \operatorname{Tr}(AK_\Delta^{(\bar{s})})\, \nu(d\bar{s}), \qquad A \in \mathfrak{T}_\Delta^{(2)} \subset \mathfrak{A}_\Delta \tag{3.2}$$

for some $\nu \in \mathscr{G}^\Phi(\Omega)$, where $K_\Delta^{(\bar{s})}$ is the operator of trace class defined by the integral kernel in (2.20). Thus we have

$$\rho_\alpha(A)=\int \operatorname{Tr}(AK_\Delta^{(\bar{s})}))\, \nu_\alpha(d\bar{s}), \qquad A \in \mathfrak{T}_\Delta^{(2)}$$

We need that the relation obtained from (3.2) by replacing ρ and ν with $\hat{\rho}$ and $\hat{\nu}$, respectively, holds. Let the interaction Φ be of finite range with the interaction range R. For given Δ choose a $\Lambda \in \mathscr{C}$ such that $\{i \in \mathbf{Z}^\nu \mid \operatorname{dist}(i, \Delta) \leqslant R\} \subset \Lambda$. For given $A \in \mathfrak{T}_\Delta^{(2)}$ we use the abbreviated notation

$$F_A(\bar{s})=\operatorname{Tr}(AK_\Lambda^{(\bar{s})})$$

In the appendix we show that one can choose $\delta > 0$ smaller than the constant $\bar{A} > 0$ appearing in Lemma A.2 such that for sufficiently large Λ the bound

$$|F_A(\bar{s})| \leqslant D(\Lambda') \prod_{\Lambda' \setminus \Lambda} e^{\delta \bar{s}_i^2} \tag{3.3}$$

holds, where $\Lambda' \supset \Lambda$ with $\text{dist}(\Lambda, \Lambda'^c) = R$. Denote by $\mu(ds)$ the finite measure $\exp(-\bar{A}s^2)\,\lambda(ds)$. Then F_A belongs to $L^1(\Omega_{\Lambda'}, \mu(ds_{\Lambda'}))$ for any $A \in \mathfrak{T}_\Delta^{(2)}$. By Lemma A.2 each Gibbs measure defines a state on $L^1(\Omega_{\Lambda'}, \mu)$. Thus each net (ν_α) of Gibbs measures has a convergent subnet which converges to a state $\tilde{\nu}$ of $L^1(\Omega_{\Lambda'}, \mu)$. Let (ν_α) be the net corresponding to the convergent net (ρ_α) of Gibbs states on $\mathfrak{A}$. As before let $\hat{\nu}$ be the limiting Gibbs measure (in the local convergence topology). Since $\tilde{\nu} = \hat{\nu}$ on the set of bounded functions on $\Omega_{\Lambda'}$, $\tilde{\nu} = \hat{\nu}$ on $L^1(\Omega_{\Lambda'}, \mu)$, and so

$$\lim_\alpha \int \nu_\alpha(d\bar{s})\, F_A(\bar{s}) = \int \hat{\nu}(d\bar{s})\, F_A(\bar{s}), \qquad A \in \mathfrak{T}_\Delta^{(2)}$$

This implies that

$$\lim_\alpha \rho_\alpha(A) = \int \hat{\nu}(d\bar{s})\, F_A(\bar{s})$$

$$= \int \hat{\nu}(d\bar{s})\, \text{Tr}(A K_\Delta^{(\bar{s})})$$

for any $A \in \mathfrak{T}_\Delta^{(2)}$. Here we have used the Gibbs conditions to get the second equality. This completes the proof. ∎

4. UNIQUENESS OF GIBBS STATES AND PHASE TRANSITIONS

After establishing the structure of the space of Gibbs states, it may be worth commenting on the uniqueness of Gibbs states and the possibility of the first-order phase transitions. We now introduce the inverse temperature $\beta = 1/kT$ in the notation in Section 2. For $x, y \in \mathbf{R}^d$ denote by $W_{x,y}^{(\beta)}$ the set of continuous paths ω: $[0, \beta] \to \mathbf{R}^d$ with $\omega(0) = x$, $\omega(\beta) = y$, and $P_{x,y}^{(\beta)}$ the conditional Wiener measure on $W_{x,y}^{(\beta)}$:

$$P_{x,y}^{(\beta)}(W_{x,y}^{(\beta)}) = (2\pi\beta)^{-d/2} \exp\left(-\frac{1}{2\beta}|x-y|^2\right)$$

We replace P and W by $P^{(\beta)}$ and $W^{(\beta)}$, respectively, in Section 2, and use the notation

$$\Omega^{(\beta)} = (\mathbf{R}^d \times W^{(\beta)})^{\mathbf{Z}^\nu}$$

and

$$\Phi_\Delta^{(\beta)}(s_\Delta) = \int_0^\beta \Phi_\Delta(s_\Delta(t))\, dt$$

$\mathscr{F}_\Lambda^{(\beta)}$, $\Lambda \in \mathscr{C}$, and $\mathscr{F}^{(\beta)}$ are the corresponding σ-algebras on $\Omega^{(\beta)}$. Denote by $\mathscr{G}^\Phi(\Omega^{(\beta)})$ the space of the Gibbs measures on $(\Omega^{(\beta)}, \mathscr{F}^{(\beta)})$ and $\mathscr{G}^{\Phi,\beta}(\mathfrak{A})$ the space of the Gibbs states on $\mathfrak{A}$.

The following properties on the uniqueness of Gibbs states should hold:

Conjecture 4.1. Under Assumption 2.1 the following results hold.

(a) For $\nu \geqslant 2$, there exists $\beta_0 > 0$ such that for any $0 < \beta < \beta_0$, $\mathscr{G}^{\Phi,\beta}(\mathfrak{A})$ is a singleton.

(b) For $\nu = 1$, $\mathscr{G}^{\Phi,\beta}(\mathfrak{A})$ is a singleton for any $\beta > 0$.

It should be possible to prove the above conjecture by developing a cluster expansion method for any Gibbs state. In refs. 20 and 17, cluster expansions for zero b.c. have been developed to show the clustering property for high temperatures and for the dimension $\nu = 1$, respectively. We plan to develop an expansion method for any Gibbs state in the near future.

We next give a brief discussion on the existence of phase transitions for lattice boson systems. For the details, we refer to ref. 12 and references therein. Consider the interaction of the form[12]

$$V(x_\Lambda) = \sum_{i \in \Lambda} V^{(1)}(x_i) - J \sum_{\substack{i,j \in \Lambda \\ |i-j| = 1}} x_i x_j \tag{4.1}$$

where $V^{(1)}$ is the one-particle interaction satisfying the following conditions: (a) $V^{(1)} \in C^\infty(\mathbf{R})$ and there exist $a > 0$, $b \in \mathbf{R}$ such that $V^{(1)}(x) \geqslant ax^2 + b$, (b) $a > J\nu$, (c) $V^{(1)}(x) = V^{(1)}(-x)$, and (d) $V^{(1)}(x)$ has global nondegenerate minima at $x = \pm q_0$, $q_0 > 0$. For $N \in \mathbf{N}$ denote by T_N the factor group $\mathbf{Z}^\nu/(N\mathbf{Z}^\nu)$, i.e., T_N is a discrete torus. Let $\omega_N^{(p)}$ be the local Gibbs states with periodic boundary conditions[11,12] and $\omega^{(p)}$ an infinite-volume limit of $\omega_N^{(p)}$, $N \in \mathbf{N}$. Define a long-range order parameter by

$$P(\beta) = \lim_{N \to \infty} \omega_N^{(p)}\left(\left(|T_N|^{-1} \sum_{i \in T_N} x_i\right)^2\right)$$

We write that

$$I_\nu = (2\pi)^{-\nu} \int_{(0,2\pi)^\nu} \left(\nu - \sum_{j=1}^{\nu} \cos p_j\right)^{-1/2} dp$$

Assume that the inequality $I_\nu(2J)^{-1/2} < q_0^2$ holds.

Theorem 4.2.[12] Let $\nu \geqslant 3$. For any interaction satisfying the conditions listed above there exists $\beta_0 > 0$ such that $P(\beta) > 0$ for any $\beta > \beta_0$.

Conjecture 4.3. Under the assumptions as in Theorem 4.2, there exists a $\beta_0 > 0$ such that $\mathrm{card}(\mathscr{G}^{\Phi,\beta}(\mathfrak{A})) = 1$ if $\beta < \beta_0$ and $\mathrm{card}(\mathscr{G}^{\Phi,\beta}(\mathfrak{A})) \geqslant 2$ if $\beta > \beta_0$.

Using the method in Section 3, it may be possible to show that any periodic state is a Gibbs state in the sense of Definition 2.8. If Conjecture 4.1 holds, then it follows that Conjecture 4.3 also holds.

The proof of Theorem 4.2 relies on the method of the infrared bounds,[8] and so the result can be extended to systems having continuous symmetry. In ref. 2 the phase diagram for a class of classical unbounded spin models has been constructed by generalizing the Pirogov–Sinai theory of phase transitions.[22] With appropriate modifications one should be able to extend the results in ref. 2 to the quantum situation.

5. DISCUSSION: OPEN PROBLEMS

As mentioned in Section 2, there are several problems to which we would like to have the answers. The local density matrix K_Λ in Definition 2.8 should be positive for any Gibbs state. We believe that the answer must follow from the definitions of Gibbs measures and the conditional reduced density matrix in (2.20). See also the discussion in the Remark below Definition 2.8.

It would be important to construct the dynamical system for the model to investigate the mathematical structure in more detail. If any Gibbs state is a modular state,[3] one has the modular automorphisms and so a dynamics. See the Remark below Theorem 2.9. Thus we would like to know whether any Gibbs state is a modular state or not. On the other hand, one may use the Green's function method[3,19,20] to construct a dynamical system at least for any infinite-volume-limit Gibbs state with constant boundary conditions. Let $(\mathfrak{H}_\rho, \pi_\rho, \Omega_\rho)$ be the cyclic representation with respect to a given Gibbs state $\rho \in \mathscr{G}^\Phi(\mathfrak{A})$. It will be nice if for any $\rho \in \mathscr{G}^\Phi(\mathfrak{A})$ there is a direct way to construct the time evolutions on $\pi_\rho(\mathfrak{A})''$ leaving Ω_ρ invariant.

APPENDIX

As stated at the beginning of Section 3, the main methods needed to prove Propositions 2.5 and 2.10 have been used in refs. 14, 17, 20, and 27. We remark that the estimates are similar to those in the propositions obtained in refs. 17 and 20.

We note that there are $r > 0$ and $\kappa > 0$ such that for all $\Lambda \in \mathscr{C}$

$$\int_{\Sigma^\Lambda} \left[\prod_{i \in \Lambda} \lambda(ds_i) \right] \exp[-V(s_\Lambda)] > \kappa^{-|\Lambda|} \tag{A.1}$$

where $\Sigma \equiv \{s \in S: |s(t)| \leqslant r,\ 0 \leqslant t \leqslant 1\}$. This is because, using Assumption 2.1(c) and (d), we have

$$V(s_\Lambda) \leqslant \sum_{i \in \Lambda} V(s_i) + M \sum_{i \in \Lambda} s_i^2$$

where $M = \sum_{j \in \mathbf{Z}^\nu} \Psi(|j|)$. The bound (A.1) now follows from (2.11).

Given $\alpha > 0$, we can choose an integer $P_0 > 0$ and for each $n \geqslant P_0$ an integer $l_n > 0$ such that

$$\left| \frac{l_{n+1}}{l_n} - (1 + 2\alpha) \right| < \alpha$$

Let $[n] = \{i \in \mathbf{Z}^\nu: |i| \leqslant l_n\}$ and $V_n = (2l_n + 1)^\nu$. The following is Proposition 2.1 of ref. 27.

Proposition A.1. Let $\varepsilon > 0$ and $c \geqslant 0$ be given, and let Ψ be a decreasing positive function on the natural integers such as given in Assumption 2.1(d). If α is sufficiently small one can choose an increasing sequence (ψ_n) such that $\psi_n \geqslant 1$, $\psi_n \to \infty$, and fix $P > P_0$ so that the following is true.

Let $n(\cdot)$ be a function from $\mathbf{Z}^\nu$ to the reals $\geqslant 0$. Suppose that there exists q such that $q \geqslant P$ and q is the largest integer for which

$$\sum_{i \in [q]} n(i)^2 \geqslant \psi_q V_q$$

Then the bound

$$\sum_{i \in [q+1]} c + \sum_{i \in [q+1]} \sum_{j \notin [q+1]} \Psi(|i-j|)\, \tfrac{1}{2}[n(i)^2 + n(j)^2] \leqslant \varepsilon \sum_{i \in [q+1]} n(i)^2$$

holds.

Proof of Proposition 2.5. We may assume the origin lies in Δ. Suppose $\bar{s} \in \mathfrak{S}$ is such that

$$\sum_{i \in [n]^*} \bar{s}_i^2 \leqslant N^2 V_n \tag{A.2}$$

for all $n \in \mathrm{N}$ and here $[n]^*$ means any translation of $[n]$ coming from a translation of the origin with Δ. We prove the proposition by using the method similar to that used in the proof of Theorem 4.1 of ref. 14. Following refs. 26 and 27 we split the configurations. For fixed $P \geqslant P_0$, put

$$\begin{aligned}
\mathfrak{R}_0 &= \left\{ s \in \Omega \,\middle|\, \sum_{i \in [q]} s_i^2 < \psi_q V_q, \forall q \geqslant P \right\} \\
\mathfrak{R}_q &= \left\{ s \in \Omega \,\middle|\, \sum_{i \in [q]} s_i^2 \geqslant \psi_q V_q \text{ and } \sum_{i \in [l]} s_i^2 < \psi_l V_l, \forall l > q \right\} \\
\mathfrak{R} &= \mathfrak{R}_0 \cup \left(\bigcup_{q \geqslant P} \mathfrak{R}_q \right)
\end{aligned} \tag{A.3}$$

From the definition $\mathfrak{S}$ in (2.16) one has $\mathfrak{S} \subset \mathfrak{R}$.

We decompose $g_\Lambda(s_\Delta \mid \bar{s})$ into two parts

$$g_\Lambda(s_\Delta \mid \bar{s}) = \rho' + \rho''$$

Here ρ' is the contribution of those $s_\Lambda \bar{s}_{\Lambda^c} \in \mathfrak{R}_0$ and ρ'' the contribution of those $s_\Lambda \bar{s}_{\Lambda^c} \in \bigcup_{q \geqslant P} \mathfrak{R}_q$.

We choose the function ψ in Proposition A.1 as (see also ref. 14)

$$\psi(r) = b \log_+ r, \qquad \log_+ r = \max\{1, \log r\} \tag{A.4}$$

with b sufficiently large and fixed. We choose $\Lambda(\Delta, \bar{s}) \supset \Delta$ so large that if the box $[q]^*$ is not contained in $\Lambda(\Delta, \bar{s})$, then $N^2 \leqslant \log l_q$. For ρ' we can proceed as in ref. 27. ρ'' is expressed as a sum over $q \geqslant P$. Still no difference arises when the cube $[q]$ (centered somewhere in Δ) is in Λ. For other cases, in ref. 27 there appears a factor $\exp[-V(s_\Lambda)]$. Therefore we lack a factor $\exp[-\sum_{i \in [q] \setminus \Lambda} (As_i^2 - c)]$. Using Assumption 2.1(c) and (d), we obtain that

$$V(s_\Lambda) + W(s_\Lambda, s_{[q] \setminus \Lambda}) + \sum_{i \in [q] \setminus \Lambda} (As_i^2 - c) \geqslant \tfrac{1}{2} \sum_{i \in [q]} (As_i^2 - c) \tag{A.5}$$

and for any $s \in \mathfrak{S}_N$

$$\begin{aligned}
\sum_{i \in [q] \setminus \Lambda} (As_i^2 - c) &\leqslant AN^2 V_{q+1} + cV_q \\
&\leqslant A \log(l_{q+1})\, V_{q+1} + cV_q
\end{aligned} \tag{A.6}$$

From the above bounds one has that

$$\begin{aligned}&\exp[-V(s_\Lambda)-W(s_\Lambda, s_{[q]\setminus\Lambda})]\\&\quad\leqslant \exp\left[-\frac{1}{2}\sum_{i\in[q]}(As_i^2-c)\right]\exp\left[\frac{A}{b}\psi_{q+1}V_{q+1}+cV_q\right]\end{aligned}$$

Following ref. 27 [and also (A4.6) of ref. 14], the following bound holds:

$$\begin{aligned}\rho''&\leqslant \exp\left(-\sum_{i\in\Delta}A^*s_i^2\right)\\&\quad\times\left[\sum_{q\geqslant P}\exp\left(-C''\psi_{q+1}V_{q+1}+D''V_{q+1}+\frac{A}{b}\psi_{q+1}V_{q+1}\right)\right]\end{aligned}$$

We choose b large enough so that $(A/b)<C''$, which guarantees the summability of (A.5). This completes the proof of the proposition. ∎

In the remainder of the appendix we use the notation

$$g_\Lambda(s_\Lambda \mid \bar{s})=Z_\Lambda(\bar{s})^{-1}\exp[-V(s_\Lambda)-W(s_\Lambda, \bar{s}_{\Lambda^c})] \tag{A.7}$$

for any $\Lambda\in\mathscr{C}$. See also the notation in (2.19).

We next show the regularity of the Gibbs measures on $(\Omega, \mathscr{F})$.

Lemma A.2. There exist positive constants $\bar{A}>0$, $\delta>0$ such that the bounds

$$g(s_\Delta \mid \nu)\leqslant\exp\left[-\sum_{i\in\Delta}(\bar{A}s_i^2-\delta)\right]$$

hold uniformly in $\nu\in\mathscr{G}^\Phi(\Omega)$, where $g(s_\Delta \mid \nu)$ is defined in Definition 2.2.

Proof. Let $\nu\in\mathscr{G}^\Phi(\Omega)$ and suppose $A\in\mathscr{F}_\Delta$. Using the equilibrium conditions, the decomposition of configurations in (A.3), and the temperedness of ν, one obtains that

$$\begin{aligned}\nu(A)&=\nu(A\cap\mathfrak{R})\\&\leqslant\nu(A\cap\mathfrak{R}_0)+\sum_{q\geqslant P}\nu(A\cap\mathfrak{R}_q)\\&=\int\nu(d\bar{s})\int\lambda(ds_{\Lambda_P})\,1_A(s_\Delta)\,1_{\mathfrak{R}_0}(s_{\Lambda_P}\bar{s})\,g_{\Lambda_P}(s_{\Lambda_P}\mid\bar{s})\\&\quad+\sum_{q\geqslant P}\int\nu(d\bar{s})\int\lambda(ds_{\Lambda_q})\,1_A(s_\Delta)\,1_{\mathfrak{R}_q}(s_{\Lambda_q}\bar{s})\,g_{\Lambda_q}(s_{\Lambda_q}\mid\bar{s})\\&=\rho'+\rho''\end{aligned} \tag{A.8}$$

In the above we choose Λ_q such that $\Delta \subset \Lambda_q$ and $[q+1] \subset \Lambda_q$ for each $q \geqslant P$. Since $[q+1] \subset \Lambda_q$, there is no lack of factors appearing in the proof of Proposition 2.5. Thus the method used in ref. 27 can be applied directly to get the uniform bounds in the lemma. ∎

Before proving Proposition 2.10, we state a lemma concerned on the Wiener measure.

Lemma A.3 (Ref. 3, Theorem 6.3.8). Let $R > 0$ be given and $D \subset \mathbf{R}^d$ be a bounded convex domain, whose boundary ∂D is a C^3-surface of mean curvature less than $1/R$. Then there exist positive numbers c', d', and e' such that the following inequality holds

$$0 \leqslant \int P_{x,y}(d\hat{s})\{1 - 1_D(\hat{s})\}$$

$$\leqslant \exp\{-c'd(x, \partial D)^2 - d'd(y, \partial D)^2 + e'\}$$

for any $x, y \in D$, where 1_D is the characteristic function of the set $\{\hat{s}: \hat{s}(t) \in D,\ 0 \leqslant t \leqslant 1\}$ and $d(x, \partial D)(d(y, \partial D))$ is the distance from $x(y)$ to ∂D.

Proof of Proposition 2.10. Let $\nu \in \mathscr{G}^{\Phi}(\Omega)$ be given. For given $\Delta \subset \mathbf{Z}^\nu$ and x_Δ and y_Δ one has

$$K_\Delta(x_\Delta, y_\Delta) = \int \nu(d\bar{s})\, k_\Delta(x_\Delta, y_\Delta; \bar{s})$$

$$= \int \nu(d\bar{s}) \int P_{x_\Delta, y_\Delta}(d\hat{s}_\Delta)\, g_\Delta(\hat{s}_\Delta \mid \bar{s}) \tag{A.9}$$

One then uses the decomposition in (A.3), the Fubini theorem, and equilibrium conditions (to L^1-funtions) to obtain

$$\begin{aligned} K_\Delta(x_\Delta, y_\Delta) = &\int P_{x_\Delta, y_\Delta}(d\hat{s}_\Delta) \int \nu(d\bar{s}) \int \lambda(ds_{\Lambda_P\setminus\Delta}) \\ &\times \{1_{\mathfrak{R}_0}(\hat{s}_\Delta s_{\Lambda_P\setminus\Delta} \bar{s}_{\Lambda_P^c})\, g_{\Lambda_P}(\hat{s}_\Delta s_{\Lambda_P\setminus\Delta} \mid \bar{s})\} \\ &+ \sum_{q \geqslant P} \int P_{x_\Delta, y_\Delta}(d\hat{s}_\Delta) \int \nu(d\bar{s}) \int \lambda(ds_{\Lambda_q\setminus\Delta}) \\ &\times \{1_{\mathfrak{R}_q}(\hat{s}_\Delta s_{\Lambda_q\setminus\Delta} \bar{s}_{\Lambda_q^c})\, g_{\Lambda_q}(\hat{s}_\Delta s_{\Lambda_q\setminus\Delta} \mid \bar{s})\} \end{aligned} \tag{A.10}$$

Although $\hat{s}_\Delta$ are not loops ($x_\Delta \neq y_\Delta$ in general), we could do the same as in Lemma A.2 to show that

$$K_\Delta(x_\Delta, y_\Delta) \leqslant \int \left[\prod_{i \in \Delta} P_{x_i, y_i}(d\hat{s}_i)\right] \exp\left[-\sum_{i \in \Lambda} (A^* \hat{s}_i^2 - \delta)\right] \tag{A.11}$$

Let us fix a number $R>0$ and $i\in\Lambda$, and suppose that $2R<\min\{|x_i|, |y_i|\}$. Define the domain D_i such that

$$D_i=\bigcup_{t\in[0,1]} B_i(t) \tag{A.12}$$

where $B_i(t)$ is the ball centered at $x_i+t(y_i-x_i)$ with radius $|x_i+t(y_i-x_i)|/2$. Obviously D_i satisfies the conditions in Lemma A.3. Now we split the integration

$$\int P_{x_i,y_i}(d\hat{s}_i)\exp(-A^*\hat{s}_i^2+\delta)=\int_{\text{in}}+\int_{\text{out}}$$

where $\int_{\text{in}}$ is the contribution of the paths from $\{\hat{s}_i: \hat{s}_i(t)\in D_i, 0\leqslant t\leqslant 1\}$ and $\int_{\text{out}}$ is that of the remaining paths. By Lemma A.3, $\int_{\text{out}}$ is bounded as

$$\int_{\text{out}}\leqslant \exp\delta\cdot\exp\left(-c'\frac{x_i^2}{4}-d'\frac{y_i^2}{4}+e'\right) \tag{A.13}$$

For $\int_{\text{in}}$ (without loss of generality we may assume $|x_i|\leqslant|y_i|$) we have $\hat{s}_i^2\geqslant\frac{1}{4}x_i^2$ and consider two cases: (a) $2|x_i|\leqslant|y_i|$ and (b) $2|x_i|>|y_i|$.

For (a),

$$\begin{aligned}\int_{\text{in}}&\leqslant\exp\left(-\frac{1}{4}A^*x_i^2+\delta\right)(2\pi)^{-d/2}\exp\left(-\frac{1}{2}|x_i-y_i|^2\right)\\&\leqslant\exp\left(-\frac{1}{4}A^*x_i^2+\delta'\right)\exp\left(-\frac{y_i^2}{8}\right)\end{aligned} \tag{A.14}$$

and for (b),

$$\begin{aligned}\int_{\text{in}}&\leqslant\exp\left(-\frac{1}{4}A^*x_i^2+\delta\right)\\&\leqslant\exp\left(-\frac{1}{8}A^*x_i^2-\frac{1}{32}A^*y_i^2+\delta\right)\end{aligned} \tag{A.15}$$

If x_i and/or y_i are near the origin, the terms $-x_i^2$ and $-y_i^2$ may be introduced at our disposal and the terms x_i^2 and y_i^2 can be absorbed to the constant term. Since the above argument can be applied to each $i\in\Lambda$, the bound for $K_\Delta(x_\Delta, y_\Delta)$ follows from Eqs. (A.13), (A.14), and (A.15).

The second part of the proposition follows from the method used in the proof of Theorem 1(d) of ref. 17. That is, if the interaction is of polynomial type, then differentiations of the function $K_\Delta(x_\Delta, y_\Delta)$ result in multiplications with "polynomial powers." Hence the second inequality in

Proposition 2.10 follows from the first one. For the details we refer to ref. 17. ▮

Proof of Bound (3.3). Let $A \in \mathfrak{T}_{\Delta}^{(2)}$ and h_A be the integral kernel function of A. Then h_A is square-integrable. By (A.9) and Fubini theorem

$$\mathrm{Tr}(AK_{\Lambda}^{(\bar{s})}) = \int dx_{\Delta} \int dy_{\Delta}\, h_A(y_{\Delta}, x_{\Delta}) \int P_{x_{\Delta}, y_{\Delta}}(d\hat{s}_{\Delta}) \int \lambda(ds_{\Lambda\setminus\Delta})$$
$$\times \int \nu(d\bar{s})\, g_{\Lambda}(\hat{s}_{\Delta} s_{\Lambda\setminus\Delta} \mid \bar{s})$$

Here we have used the notation (A.7). Let $\Lambda' = \{i \mid \mathrm{dist}(i, \Lambda) \leqslant R\}$. Then it can be written that

$$\int \lambda(ds_{\Lambda\setminus\Delta}) \int \nu(d\bar{s})\, g_{\Lambda}(\hat{s}_{\Delta} s_{\Lambda\setminus\Delta} \mid \bar{s})$$
$$= \int \nu(d\bar{s}) \int \lambda(ds_{\Lambda\setminus\Delta})\, Z(\bar{s}_{\Lambda'})^{-1}$$
$$\times \exp[-V(\hat{s}_{\Delta} s_{\Lambda\setminus\Delta}) - W(\hat{s}_{\Delta} s_{\Lambda\setminus\Delta}, \bar{s}_{\Lambda'\setminus\Lambda})] \qquad \text{(A.16)}$$

As in the proof of Proposition 2.5, we write the above as a sum of ρ' and ρ''. As before we lack a factor $\exp[-\sum_{i \in [q]\setminus\Lambda} (As_i^2 - c)]$. Choose a sufficiently small $\delta > 0$ (e.g., $0 < \delta < A/4$) and substitute

$$1 = \prod_{i \in \Lambda'\setminus\Lambda} \exp(-\delta \bar{s}_i^2) \prod_{j \in \Lambda'\setminus\Lambda} \exp(\delta \bar{s}_i^2)$$

into the right-hand side of (A.16). With the new superstable constant δ one carry out the estimate as in ref. 27. Thus there exist $\gamma > 0$ and c such that (A.16) is bounded by

$$\left\{\prod_{i \in \Delta} \exp[-(\gamma \hat{s}_i^2 - c)]\right\} \left[\prod_{j \in \Lambda'\setminus\Lambda} \exp(\delta \bar{s}_j^2)\right]$$

Performing the remaining integration, one gets the bound. ▮

Finally we collect the result used in the proof of Theorem 2.7. Put

$$X(a) = \{s \in \Omega \mid s_i^2 < a \log |i|,\ |i| \geqslant 1\}, \qquad a > 0$$

The following is the result corresponding to Theorem 4.1 of ref. 14.

Lemma A.4. There exist $\bar{A}>0$ and $\delta>0$ such that the following result holds: For every $\Lambda \in \mathscr{C}$ there is $\Lambda(\Delta)$ such that for $\bar{s} \in X(a)$ the bound

$$g_\Lambda(s_\Delta \mid \bar{s}) \leqslant \exp\left[-\sum_{i \in \Delta} (\bar{A}s_i^2 - \delta)\right]$$

holds.

Proof. We remark that by Assumption 2.1(d) the condition in Hypothesis 4.1 of ref. 15 is satisfied. Thus a direct application of the method used in the proof of Theorem 4.1 of ref. 14 gives the result. ∎

ACKNOWLEDGMENTS

The authors would like to thank the referees for their comments on the manuscript, which enabled us to improve it considerably. This research was supported in part by the Basic Science Research Program, Ministry of Education, 1994.

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